

ETERNAL SOLUTIONS OF THE BURGERS EQUATION

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Abstract: Solutions that satisfy classically the Burgers equation except, perhaps, on a closed set \mathcal{S} of the plane of potential singularities whose Hausdorff 1-measure is zero, $H^1(\mathcal{S}) = 0$, are necessarily identically constant. We show this under the additional hypothesis that \mathcal{S} is a subset of a finite union of smooth graphs.

1. Introduction

The result stated in the Abstract is based on a simple change of variables which transforms the Burgers equation to the eikonal equation. Utilizing then a theorem of Caffarelli and Crandall [1] on the eikonal equation we can conclude. So it is completely elementary, and our only excuse for writing it down is that it concerns the Burgers equation, which, in spite of its simplicity, pervades the theory of hyperbolic conservation laws [2],[3]. We believe that the restriction that \mathcal{S} lies in a finite union of graphs is not necessary. Our result implies in particular that there are no entire solutions with \mathcal{S} a singleton or a Cantor set arranged on a graph.

2. Statements and Proofs

Theorem 1

Let $h(x,t)$ be a measurable function on \mathcal{R}^2 and suppose that, except on a closed set $\mathcal{S} \subset \mathcal{R}^2$ of potential singularities, $\frac{\partial h}{\partial t}$ exist, $x \rightarrow \frac{\partial h}{\partial x}(x,t)$ is continuous at $(x,t) \in \mathcal{R}^2 \setminus \mathcal{S}$ and h solves the Burgers equation pointwise

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} h^2 \right) = 0 \quad , \text{ on } \mathcal{R}^2 \setminus \mathcal{S} \quad (1)$$

If $\mathcal{S} \subset \cup_{i=1}^n \Gamma_i$, where $\Gamma_i := \{(x,t) | t = p_i(x), x \in \mathcal{R}\}$, where $p_i(x)$ is differentiable, $p_1(x) < p_2(x) < \dots < p_n(x) \forall x \in \mathcal{R}$ and $H^1(\mathcal{S}) = 0$, then h is identically constant.

Proof

For the convenience of the reader we begin by giving the proof in the simpler case that $\mathcal{S} \subset \Gamma := \{(x,t) | t = p(x), x \in \mathcal{R}\}$, $p(x)$ differentiable and $0 < p(x) < 1 \forall x \in \mathcal{R}$. Set $\Omega^+ = \{(x,t) \in \mathcal{R}^2 | t \leq p(x)\}$, $\Omega^- = \{(x,t) \in \mathcal{R}^2 | t \geq p(x)\}$ Define for $(x,t) \in \Omega^+$ ($t \leq p(x)$)

$$f_+(x,t) = \int_0^t \frac{ds}{\sqrt{h^2(x,s) + 1}} + g_+(x) \quad (2)$$

where $g_+(x) = \int_0^x \frac{h(u,0)du}{\sqrt{h^2(u,0) + 1}}$
and for $(x,t) \in \Omega^-$ ($t \geq p(x)$)

$$f_-(x,t) = \int_1^t \frac{ds}{\sqrt{h^2(x,s) + 1}} + g_-(x) \quad (3)$$

where $g_-(x) = \int_0^x \frac{h(u,1)du}{\sqrt{h^2(u,1)+1}}$

We begin with $f_+(x, t)$ for $t < p(x)$, (x, t) otherwise arbitrary

$$\frac{\partial}{\partial x} f_+(x, t) = \int_0^t \frac{-h(x, s) \frac{\partial h}{\partial x}(x, s)}{(\sqrt{h^2(x, s) + 1})^3} ds + g_+'(x) \quad (4)$$

Here we have utilized the hypothesis that $\frac{\partial h}{\partial x}$ is continuous at $(x, t) \in \mathcal{R}^2 \setminus \mathcal{S}$ which allows the differentiation under the integral sign, by Lebesgue's dominated covergens theorem

$$\frac{\partial}{\partial x} \int_0^t Q(x, s) ds = \int_0^t Q_x(x, s) ds, \quad |Q_x| \leq (const.) |h_x|, \quad \int_0^t |h_x(x, s)| ds < \infty.$$

The 2^{nd} term in (4) is even easier to justify. Thus utilizing equation (1)

$$\frac{\partial}{\partial x} f_+(x, t) = \int_0^t \frac{h_s(x, s)}{(\sqrt{h^2(x, s) + 1})^3} ds + \frac{h(x, 0)}{\sqrt{h^2(x, 0) + 1}} \quad (5)$$

$$= \frac{h(x, t)}{\sqrt{h^2(x, t) + 1}}. \quad (6)$$

Next we will be extending the computation all the way to the graph Γ :

$$\frac{\partial f_+}{\partial x}(x, p(x)) = \frac{h(x, p(x))}{\sqrt{h^2(x, p(x)) + 1}}, \quad \text{for } (x, p(x)) \notin \mathcal{S} \quad (7)$$

For this purpose we notice that at such x the function $h_x(x, s)$ is continuous for all s . Hence the same argument applies and we obtain (7). Differentiating $f_+(x, t)$ in t is easy and holds quite generally,

$$\frac{\partial}{\partial t} f_+(x, t) = \frac{1}{\sqrt{h^2(x, t) + 1}}, \quad \frac{\partial}{\partial t} f_+(x, p(x)) = \frac{1}{\sqrt{h^2(x, p(x)) + 1}} \quad (8)$$

Analogously we argue for $f_-(x, t)$ and thus we obtain

$$\left(\frac{\partial f_+}{\partial t}(x, t)\right)^2 + \left(\frac{\partial f_+}{\partial x}(x, t)\right)^2 = 1 \quad \text{in } \Omega^+ \setminus \mathcal{S}, \quad (9)$$

$$\left(\frac{\partial f_-}{\partial t}(x, t)\right)^2 + \left(\frac{\partial f_-}{\partial x}(x, t)\right)^2 = 1 \quad \text{in } \Omega^- \setminus \mathcal{S} \quad (10)$$

Next we observe that for $t < p(x)$

$$f_+(x, t) - f_+(x, p(x)) = - \int_t^{p(x)} \frac{ds}{\sqrt{h^2(x, s) + 1}} \quad (11)$$

Similarly for $\tau > p(x)$

$$f_-(x, \tau) - f_-(x, p(x)) = \int_{p(x)}^{\tau} \frac{ds}{\sqrt{h^2(x, s) + 1}} \quad (12)$$

Therefore

$$\int_t^\tau \frac{ds}{\sqrt{h^2(x, s) + 1}} = \Delta(x) - (f_+(x, t) - f_-(x, \tau)) \quad (13)$$

$t < p(x) < \tau$, $\Delta(x) := f_+(x, p(x)) - f_-(x, p(x))$, $\forall x \in \mathcal{R}$

On the other hand, from (7), (8) and their analogs for $f_-(x, t)$ we obtain
for $(x, p(x)) \notin \mathcal{S}$

$$\frac{\partial f_+}{\partial x}(x, p(x)) = \frac{\partial f_-}{\partial x}(x, p(x)) \quad (14)$$

$$\frac{\partial f_+}{\partial t}(x, p(x)) = \frac{\partial f_-}{\partial t}(x, p(x)) \quad (15)$$

$\Gamma \setminus \mathcal{S}$ is open in Γ , and so is its projection on the x-axis,

$\pi_x(\Gamma \setminus \mathcal{S}) = \bigcup_i (a_i, b_i) =: \mathcal{O}$ Also $H^1(\mathcal{S}) = 0 \Rightarrow L^1(\pi_x(\mathcal{S})) = 0$

For $x \in \mathcal{O}$ we note that

$$\frac{d}{dx} \Delta(x) = \frac{\partial f_+}{\partial x}(x, p(x)) + \frac{\partial f_+}{\partial t}(x, p(x)) p'(x) - \left(\frac{\partial f_-}{\partial x}(x, p(x)) + \frac{\partial f_-}{\partial t}(x, p(x)) p'(x) \right) \quad (16)$$

$= 0$ (by (14), (15))

We now define

$$f(x, t) = \begin{cases} f_+(x, t), & (x, t) \in \Omega^+ \\ f_-(x, t) + \Delta(x), & (x, t) \in \Omega^- \end{cases} \quad (17)$$

We note that $f(x, t)$ is continuous on $\mathcal{R}^2 \setminus \mathcal{S}$, and the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}$ exist on $\mathcal{R}^2 \setminus \mathcal{S}$ and satisfy

$$\left(\frac{\partial f}{\partial t}(x, t) \right)^2 + \left(\frac{\partial f}{\partial x}(x, t) \right)^2 = 1 \quad \text{for } (x, t) \in \mathcal{R}^2 \setminus \mathcal{S}. \quad (18)$$

Hence by a result in [CC], f is either affine

$$f(x, t) = \alpha x + \beta t + \gamma \quad (\alpha^2 + \beta^2 = 1) \quad (19)$$

or a cone function

$$f(x, t) = c \pm \sqrt{(x - x_0)^2 + (t - t_0)^2} \quad (20)$$

In the first case gives $\frac{\partial f}{\partial t}(x, t) = \beta$ and so $h(x, t) \equiv \text{constant}$.

On the other hand (20) gives

$$\frac{\partial f}{\partial t}(x, t) = \pm \frac{t - t_0}{\sqrt{(x - x_0)^2 + (t - t_0)^2}} \quad (21)$$

$$\Leftrightarrow h(x, t) = \frac{x - x_0}{t - t_0} \quad (22)$$

which is singular on $\{t = t_0\}$, and thus is excluded by the hypothesis $H^1(\mathcal{S}) = 0$.
Therefore $h(x, t) \equiv \text{constant}$ is the only option.

Now, in the general case we indicate the necessary modifications. Since $p_i(x) < p_{i+1}(x)$, exists $a_i(x)$ differentiable : $p_i(x) < a_i(x) < p_{i+1}(x)$

$a_0(x) := p_1(x) - 1$, $a_n(x) := p_n(x) + 1$.
Set $\Omega_i^+ = \{(x, t) | t \leq p_i(x)\}$, $\Omega_i^- = \{(x, t) | t \geq p_i(x)\}$ ($i = 1, \dots, n$)
We start the process from the bottom up, so for $(x, t) \in \Omega_1^+$,

$$f_1^+(x, t) := \int_{a_0(x)}^t \frac{ds}{\sqrt{h^2(x, s) + 1}} + g_1^+(x) \quad (23)$$

where $g_1^+(x) := \int_0^x \frac{h(s, a_0(x)) + a_0'(x)}{\sqrt{h^2(s, a_0(x)) + 1}} ds$
for $(x, t) \in \Omega_i^-$,

$$f_i^-(x, t) := \int_{a_i(x)}^t \frac{ds}{\sqrt{h^2(x, s) + 1}} + g_i^-(x) \quad (24)$$

where $g_i^-(x) = \int_0^x \frac{h(s, a_i(x)) + a_i'(x)}{\sqrt{h^2(s, a_i(x)) + 1}} ds$, for $i = 1, \dots, n$
also, for $(x, t) \in \Omega_i^+$

$$f_i^+(x, t) := f_{i-1}^-(x, t) + \Delta_{i-1}(x) \quad (i = 2, \dots, n) \quad (25)$$

where $\Delta_i(x) := f_i^+(x, p_i(x)) - f_i^-(x, p_i(x))$, $i = 1, \dots, n$

$$f_i(x, t) := \begin{cases} f_i^-(x, t) + \Delta_i(x), & \text{in } \Omega_i^- \cap \Omega_{i+1}^+ \\ f_i^+(x, t), & \text{in } \Omega_i^+ \cap \Omega_{i-1}^- \end{cases}$$

and for every f_i :

$$\|\nabla f_i(x, t)\|_2 = 1, \text{ in } [(\Omega_i^- \cap \Omega_{i+1}^+) \cup (\Omega_i^+ \cap \Omega_{i-1}^-)] \setminus \mathcal{S} \quad (26)$$

On the other hand, $f_i(x, t) = f_{i-1}(x, t)$, for $(x, t) \in \Omega_i^- \cap \Omega_{i+1}^+$, by (25)

So,

$$f(x, t) := \begin{cases} f_i(x, t), & (x, t) \in \Omega_i^- \cap \Omega_{i+1}^+ \quad (i=2, \dots, n-1) \\ f_1^+(x, t), & (x, t) \in \Omega_1^+ \end{cases}$$

and $f(x, t) := f_n^-(x, t)$, $(x, t) \in \Omega_n^-$ then f satisfies:

$$\|\nabla f(x, t)\|_2 = 1, \text{ in } \mathcal{R}^2 \setminus \mathcal{S} \quad (27)$$

and we conclude. \square

Remarks

1. Note that $\Delta(x)$ is continuous for $x \in \mathcal{R}$

$$\Delta(x) = f_+(x, p(x)) - f_-(x, p(x))$$

is clearly continuous for $x \in \mathcal{O}$. Take a x_0 arbitrary and consider two sequences $\{x_n^-\}, \{x_n^+\}$ approaching x_0 from the left and right respectively. From (13) we have

$$\int_t^\tau \frac{ds}{\sqrt{h^2(x_n^+, s) + 1}} = \Delta(x_n^+) - (f_+(x_n^+, t) - f_-(x_n^+, \tau)) \quad (28)$$

$$\int_t^\tau \frac{ds}{\sqrt{h^2(x_n^-, s) + 1}} = \Delta(x_n^-) - (f_+(x_n^-, t) - f_-(x_n^-, \tau)) \quad (29)$$

We choose $t < p(x_0) < \tau$

Substracting and estimating gives

$$|\Delta(x_n^+) - \Delta(x_n^-)| \leq |f_+(x_n^+, t) - f_+(x_n^-, t)| + |f_-(x_n^+, \tau) - f_-(x_n^-, \tau)| + 2|t - \tau|.$$

$x \rightarrow \Delta(x)$ is locally bounded, hence the limits (along subsequences)

$\Delta^+ = \lim \Delta(x_n^+)$, $\Delta^- = \lim \Delta(x_n^-)$ are necessarily equal.

2. If $x \rightarrow h(x, t)$ has locally finite total variation, \mathcal{S} is discrete and

$\Delta(x) \equiv \text{const.}$

3. Since

$$\text{div}_{x,t}(\frac{1}{2}h^2, h) = 0 \quad (30)$$

and the eikonal equation

$$(\frac{\partial f}{\partial t})^2 + (\frac{\partial f}{\partial x})^2 = 1 \quad (31)$$

are invariant ,under rotations,the result above holds if $\mathcal{S} \subset \cup_{i=1}^n \Gamma_i$, where Γ_i is a graph with respect to a set of orthogonal axes in the x-t plane.

4.Note that solutions of $h_t + hh_x = 0$ and $(f_t)^2 + (f_x)^2 = 1$ linked via transformation (2) have the same characteristics.

5.Shock waves and rarefaction waves show that the hypothesis $H^1(\mathcal{S}) = 0$ can not be relaxed to $H^{1+\varepsilon}(\mathcal{S}) = 0$ for any $\varepsilon > 0$.

References

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